Error estimates for transport problems with high Péclet number using a continuous dependence assumption

Article *in* Journal of Computational and Applied Mathematics · July 2016

DOI: 10.1016/j.cam.2016.06.024

2 authors:

Erik Burman
University College London

Isaac P. Santos
Universidade Federal do Espírito Santo

Journal of Computational and Applied Mathematics 309 (2017) 267–286

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Error estimates for transport problems with high Péclet number using a continuous dependence assumption

Erik Burman [a], Isaac P. Santos [b,*]

[a] Department of Mathematics, University College London - UCL, United Kingdom

[b] Department of Applied Mathematics, Federal University of Espírito Santo, Brazil

ARTICLE INFO

Article history:
Received 11 February 2015

Keywords:
Continuous dependence
Advection–diffusion equation
Stabilized finite element method
Error estimates

ABSTRACT

In this paper we discuss the behavior of stabilized finite element methods for the transient advection–diffusion problem with dominant advection and rough data. We show that provided a certain continuous dependence result holds for the quantity of interest, independent of the Péclet number, this quantity may be computed using a stabilized finite element method in all flow regimes. As an example of a stable quantity we consider the parameterized weak norm introduced in Burman (2014). The same results may not be obtained using a standard Galerkin method. We consider the following stabilized methods: *Continuous Interior Penalty* (CIP) and *Streamline Upwind Petrov–Galerkin* (SUPG). The theoretical results are illustrated by computations on a scalar transport equation with no diffusion term, rough data and strongly varying velocity field.

1. Introduction

The numerical solution for transient convection–diffusion equations is characterized by the mesh Péclet number. It is well known that for rough data or in cases where sharp layers develop during the time evolution approximations obtained by the standard Galerkin method suffer from numerical instabilities that make the rate of convergence deteriorate. The task of designing robust a posteriori and a priori error estimates for this problem remains a challenging problem. In particular in the case of a high Péclet number and a strongly varying velocity field strong amplification of errors may occur. A recent analysis of this case was presented in [1]. There it was shown that if the error was measured in a weak norm and the velocity field had a certain scale separation property, error estimates could be obtained for problems with initial data and source term in L^2. The constant of these estimates exhibits exponential growth, but the exponential factor is proportional to the gradient of the large scales of the velocity field only. Hence fluctuations with small amplitude in the vector field do not contribute to error growth, regardless of their gradients, provided they can be dominated by the molecular diffusion.

In this paper we revisit this type of error estimates and show that the same analysis can be carried out assuming a certain type of continuous dependence on data. The result of [1] then enters our framework of an example of a stable quantity. Indeed it appears that in the high Péclet regime the continuous dependence on data for the continuous problem is inherited by the finite element method only when a stabilized method is used, and only in this case, can we obtain accurate approximate solutions of the problem independent of the mesh Péclet number. The stabilized methods considered are the Continuous Interior Penalty method (CIP) [2] and the Streamline Upwind Petrov–Galerkin (SUPG) [3,4]. In the numerical section we

* Corresponding author.
E-mail addresses: e.burman@ucl.ac.uk (E. Burman), isaac.santos@ufes.br (I.P. Santos).

http://dx.doi.org/10.1016/j.cam.2016.06.024

investigate if the error estimates remain sharp in the limit case of vanishing viscosity. Both the convergence order in various norms and the perturbation growth with respect to the variation of the velocity field are studied.

The results of this paper were inspired by the reported successful computations of averaged quantities in turbulent flows using stabilized finite element methods and adaptivity driven by the computation of sensitivities [5,6]. Although our model problem is very simple we hope that the ideas can be made to bear on more complex problems. The implication would be that in a globally ill-conditioned (or even ill-posed) problem better stability could hold for certain quantities and that these quantities may be computed using a stabilized finite element method. This program has been carried out in the ill-posed case for the linear elliptic Cauchy problem [7] using conditional stability and in parallel to the present work for the two-dimensional Navier–Stokes' equations in [8]. For other recent work on a posteriori error estimation for convection–diffusion equations we refer to [9–14].

Consider the unsteady advection–diffusion problem given by

$$\partial_t u - \mu\Delta u + \boldsymbol{\beta}\cdot\nabla u = f, \quad \text{in } \Omega\times(0,T); \tag{1}$$

$$u = 0, \quad \text{on } \partial\Omega\times(0,T); \tag{2}$$

$$u(\cdot,0) = u_0, \quad \text{in } \Omega, \tag{3}$$

where $\Omega \subset \mathbb{R}^d$, with $d \in \{2,3\}$, is an open, convex polygonal or polyhedral domain with boundary $\partial\Omega$ and outward pointing normal $\boldsymbol{\eta}_{\partial\Omega}$. We denote the space–time domain by $Q = \Omega\times I$, where $I = (0,T)$, and $T > 0$ is the final time. Also, $\boldsymbol{\beta} \in [C^0(I, W^{1,\infty}(\Omega))]^d$ is the velocity field satisfying $\nabla\cdot\boldsymbol{\beta} = 0$, $f \in L^2(Q)$ is the source/sink term, $\mu \in \mathbb{R}$ with $\mu > 0$ is the diffusivity coefficient and $u_0 \in L^2(\Omega)$ is the initial solution. We use the notation $a \preccurlyeq b \iff a \le Cb$ where $C > 0$ is a constant that does not depend on μ, h and Δt; it depends only on low order powers of T and the local mesh geometry. We will also use the notation $a \sim b$ for $a \preccurlyeq b$ and $b \preccurlyeq a$. We denote by $(\cdot,\cdot)_X$ the usual inner product in $L^2(X)$ with $X \subseteq \Omega$ and by $(\cdot,\cdot)$ if $X = \Omega$. For the space $L^2(\Omega)$ we use the usual norm, $\|\cdot\|$, and for $L^2(X)$ with $X \subset \Omega$ or $X = Q$ the norm is denoted by $\|\cdot\|_X$. The norms on $V = H^1_0(\Omega)$, $V' = H^{-1}(\Omega)$ and $L^q(I,V')$ are denoted by $\|\cdot\|_V$, $\|\cdot\|_{V'}$ and $\|\cdot\|_{L^q(I,V')}$, respectively, with $1 \le q < \infty$.

The variational formulation of problem (1)–(3) may be written, for $t \in I$, find $u \in V = H^1_0(\Omega)$ such that $u(x,0) = u_0(x)$ and

$$(\partial_t u, v) + a(u,v) = \ell(v), \quad \forall v \in V, \tag{4}$$

where

$$a(u,v) = (\mu\nabla u, \nabla v) + (\boldsymbol{\beta}\cdot\nabla u, v); \tag{5}$$

$$\ell(v) = (f, v). \tag{6}$$

The standard global regularity estimates for problem (1)–(3) depend on the parameter μ^{-1}. Consequently, they are sensitive to the variation of the diffusivity and cannot be used when the problem is advection dominated [1]. On the other hand global regularity estimates without the inverse power of μ can be obtained, assuming some more regularity of data: $f \in L^2(I; H^1_0(\Omega))$ and $u_0 \in H^1_0(\Omega)$. The constant of the stability estimate then depends on $e^{\|\nabla_s\boldsymbol{\beta}\|T}$ where ∇_s denotes the symmetric part of the gradient.

We assume that the problem is normalized so that $\|\boldsymbol{\beta}\|_{L^\infty(Q)} = 1$. In the analysis below a special role will be played by velocity fields satisfying a particular multiscale behavior that may be written as follows. There exists a decomposition of the velocity field,

$$\boldsymbol{\beta} = \overline{\boldsymbol{\beta}} + \boldsymbol{\beta}', \tag{7}$$

where $\overline{\boldsymbol{\beta}}$ is associated with the resolved scale resolution and $\boldsymbol{\beta}'$ is associated with the fine scales. Moreover, for all t, $\|\overline{\boldsymbol{\beta}}\|_{W^{1,\infty}(\Omega)} \sim 1$ and $\|\boldsymbol{\beta}'\|^2_{L^\infty(\Omega)} \sim \mu$. Under this assumption we may define a timescale for the flow relating to both the resolved scale and fine scale,

$$(\tau_F)^{-1} = \max\{\|\overline{\boldsymbol{\beta}}(t)\|_{W^{1,\infty}(\Omega)}, \|\boldsymbol{\beta}'(t)\|^2_{L^\infty(\Omega)}/\mu\} \sim 1. \tag{8}$$

Essentially we assume that the velocity field can be decomposed in a coarse scale, responsible for transport, that is slowly varying in space and a fine scale, responsible for mixing, that has small amplitude but may have very strong spatial variation. Expressed in Péclet numbers this means that the coarse scale Péclet number may be arbitrarily high, whereas the fine scale Péclet number must be of order one. We also assume that the velocity field satisfies non-penetration boundary conditions $\boldsymbol{\beta}\cdot\boldsymbol{\eta}_{\partial\Omega} = 0$.

2. Finite element approximation

Let $\mathcal{T}_h = \{K\}$ be a non-overlapping conforming, quasi uniform triangulation of the domain Ω, where $h = \max h_K$ stands for the mesh parameter with h_K the diameter of triangle $K \in \mathcal{T}_h$. For each element K we define the outward pointing normal $\boldsymbol{\eta}_{\partial K}$. The set of interior faces $\{F\}$ of $\mathcal{T}_h$ is denoted by $\mathcal{F}$ and for each $F \in \mathcal{F}$, h_F denotes its diameter and $\boldsymbol{\eta}_F$ a normal to the face the orientation of which is arbitrary but fixed. We will use the notation $\boldsymbol{\eta}$ for a function such that $\boldsymbol{\eta}|_F = \boldsymbol{\eta}_F$. We assume that the triangulation $\mathcal{T}_h$ is regular, that is, for suitable $\sigma > 0$, we have

$$\frac{h_K}{\rho_K} \leq \sigma, \quad \forall K \in \mathcal{T}_h,$$

where ρ_K is the diameter of the largest circle inscribed in K. Moreover, the following inverse inequalities are known to hold on V_h,

$$\|\nabla v_h\|_K \leq c_i h_K^{-1} \|v_h\|_K; \tag{9}$$

$$\|v_h\|_{\partial K} \leq c_t h_K^{-1/2} \|v_h\|_K; \tag{10}$$

$$h\|\Delta v_h\|_K \leq c_i \|\nabla v_h\|_K. \tag{11}$$

The elementwise and facewise L^2-norms will be defined by

$$\|v\|_{\mathcal{T}}^2 := \sum_{K \in \mathcal{T}_h} \|v\|_K^2, \qquad \|v\|_{\mathcal{F}}^2 := \sum_{F \in \mathcal{F}} \|v\|_F^2.$$

We let $V_h \subset V$ denote the standard finite element space defined by

$$V_h = \{v_h \in H_0^1(\Omega);\ v_h|_K \in \mathcal{P}_r(K),\ \forall K \in \mathcal{T}_h\},$$

with $\mathcal{P}_r(K)$ denoting the polynomials space of degree less than or equal to r on K. We denote by $\pi_h : L^2(\Omega) \longrightarrow V_h$ the L^2-projection and by $\pi_{\tilde{K}} : L^2(\Omega) \longrightarrow V_h$ the usual Clément interpolation operator [15] and we also introduce the following known inequalities,

$$h^{-1}\|v - \pi_h v\| + \|\nabla(v - \pi_h v)\| + h^{-\frac{1}{2}}\|v - \pi_h v\|_{\mathcal{F}} \leq C_h \|v\|_{H^1(\Omega)} \tag{12}$$

and

$$\|v - \pi_{\tilde{K}} v\|_K \leq C_1 h_K \|v\|_{H^1(\widetilde{K})}; \tag{13}$$

$$\|v - \pi_{\tilde{K}} v\|_{\partial K} \leq C_2 h_K^{1/2} \|v\|_{H^1(\widetilde{K})}, \tag{14}$$

where C_1 and C_2 are two positive constants that depend on the minimal angle of the elements of $\mathcal{T}_h$ and $\widetilde{K}$ denotes the sub-domain of elements sharing a common side or vertex with K. Observe that $\bigcup_{K \in \mathcal{T}_h} \widetilde{K}$ covers Ω only a finite number of times uniformly in h. The standard finite element method applied to (1)–(3) reads, for $t > 0$, find $u_h \in V_h$, such that $u_h(x, 0) = u_0(x)$ and

$$(\partial_t u_h, v_h) + a(u_h, v_h) = \ell(v_h), \quad \forall v_h \in V_h. \tag{15}$$

Since $V_h \subset V$, the exact solution u satisfies Eq. (4) for each $v = v_h \in V_h$, hence we have

$$(\partial_t u, v_h) + a(u, v_h) = \ell(v_h), \quad \forall v_h \in V_h. \tag{16}$$

By subtracting side to side (15) from (16) and defining the numerical error $e = u - u_h \in V$, we get the equation

$$(\partial_t e, v_h) + a(e, v_h) = 0, \quad \forall v_h \in V_h, \tag{17}$$

known as Galerkin orthogonality. On the other hand, taking $v \in V$, the error e satisfies the equation

$$(\partial_t e, v) + a(e, v) = \ell(v) - \Big[(\partial_t u_h, v) + a(u_h, v)\Big], \quad \forall v \in V, \tag{18}$$

so that

$$R_h(v) = \ell(v) - \Big[(\partial_t u_h, v) + a(u_h, v)\Big], \quad \forall v \in V, \tag{19}$$

denotes the weak residual. In particular, the Galerkin orthogonality property (17) implies that

$$R_h(v_h) = 0, \quad \forall v_h \in V_h. \tag{20}$$

The linearity of R_h implies that $R_h(v) = R_h(v - \pi_h v)$. For each $t \in I$ we define the weak residual by

$$R_h(v) = \ell(v - \pi_h v) - \Big[(\partial_t u_h, v - \pi_h v) + a(u_h, v - \pi_h v)\Big], \quad \forall v \in V. \tag{21}$$

3. Continuous dependence and the dual problem

Our aim is to show that the stabilized methods under consideration are robust for computations at high Péclet number. The key ingredients for doing this are:

- continuous dependence on data independent of the Péclet number;
- sufficient control of the discrete residual.

The order of convergence obtained depends on the norms used for these two factors and the a priori control of the residual in these norms given by the stabilization. We introduce the dual norm on ℓ defined by

$$\|\ell\|_{L^1(I,V')} = \int_I \|\ell\|_{V'}\,\mathrm{d}t, \tag{22}$$

where for each $t \in I$,

$$\|\ell\|_{V'} = \sup_{v\in V;\|v\|_V=1} |\ell(v)|. \tag{23}$$

Assumption 3.1 (*Continuous Dependence on Data*)**.** Let $J : V \longrightarrow \mathbb{R}$ be a functional representing some quantity of interest associated to the problem and $\Theta : [0,\infty) \to \mathbb{R}$ a continuous increasing function satisfying $\lim_{x\to 0^+} \Theta(x) = 0$. We assume that for $\ell \in L^1(I, V')$ and a sufficiently small $\varepsilon > 0$, there holds, for u the solution of (4)

$$\|\ell\|_{L^1(I,V')} \le \varepsilon \quad \text{then } |J(u)| \le \Theta(\varepsilon), \tag{24}$$

where ℓ is defined according to (4).

Assuming that the continuous problem (4) satisfies property (24), we show that this same property may be used for obtaining robust error estimates with respect to the Péclet number, of the finite element method only if a stabilized method is used. The results are demonstrated for CIP and SUPG stabilized methods.

A convenient way of expressing the continuous dependence of Assumption 3.1 is by using a dual adjoint problem. Consider the abstract problem: find $u \in V$ with $u(\cdot, 0) = 0$ such that

$$(\partial_t u, v) + a(u, v) = \ell(v), \quad \forall v \in V,\ t \in I, \tag{25}$$

where $a(\cdot,\cdot)$ is a elliptic operator. Now we introduce the following dual problem: find $\varphi \in V$ with $\varphi(\cdot, T) = \psi_T$ such that

$$(w, -\partial_t \varphi) + a(w, \varphi) = 0, \quad \forall w \in V,\ t \in I. \tag{26}$$

Suppose that the quantity of interest related to solution u is a scalar quantity expressed by

$$J(u) = (u(\cdot, T), \psi_T). \tag{27}$$

By choosing $w = u$ in (26) and using integration by parts the functional J can be rewritten by

$$J(u) = (\partial_t u, \varphi) + a(u, \varphi) = \ell(\varphi), \quad \forall u \in V,\ t \in I. \tag{28}$$

This implies

$$|J(u)| \le \|\ell\|_{L^1(I;V')} \|\varphi\|_{L^\infty(I;V)}. \tag{29}$$

For $\psi_T = \varphi(0, T) \in H_0^1(\Omega)$ we define $C_s(\psi_T) = \|\varphi\|_{L^\infty(I;V)}$. The coefficient C_s, known as *stability factor*, measures the sensitivity to discretization errors for approximate $J(u)$. Given $\ell \in L^1(I; V')$ satisfying (25) an important aspect to be evaluated is to know when the right hand side of (29) is bounded. On the other hand, if $\|\ell\|_{L^1(I;V')}$ satisfies

$$\|\ell\|_{L^1(I;V')} < \varepsilon,$$

with $\varepsilon > 0$, we can write

$$|J(u)| \le \varepsilon C_s.$$

In this case, the problem (25) satisfies the continuous dependence assumption with $\Theta(\varepsilon) = \varepsilon C_s$. Moreover we have shown that the error $e = u - u_h$, in the context of finite element method, satisfies the equation

$$(\partial_t e, \varphi) + a(e, \varphi) = \ell_e(\varphi). \tag{30}$$

The right hand side is given by

$$\ell_e(\varphi) = R_h(\varphi), \tag{31}$$

and as we will show below, this quantity can be upper bounded, independently of the diffusion coefficient, *only if a stabilized method is used*. In this case, the bound on $\ell_e(\varphi)$ is given by

$$\|\ell_e\|_{L^1(I;V')} \preccurlyeq Ch^{1/2},$$

where the constant $C = C(f, u_0, T)$ does not depend on neither the diffusion coefficient nor on special properties of the exact solution. Consequently, by using the continuous dependence assumption (Assumption 3.1), and the control of the weak residual we obtain the bound

$$|J(u - u_h)| \leq \Theta(Ch^{1/2}).$$

This quantity can be some norm of the error or the error in an average over some subset of the domain or even the error at some point of Ω.

3.1. Example of a stable quantity: the regularized error

It is not obvious to find quantities for which Assumption 3.1 holds, but one example is the estimate on the regularized error studied in [1]. The idea is to apply a differential filter to the error and use the smoothed error as data in the dual problem. Using an error quantity associated to this regularized error the required stability of the dual problem may be shown.

The regularized error is obtained by using a differential filter through elliptic boundary value problem: find $\tilde{e}$ such that

$$\begin{aligned} -\delta^2 \Delta \tilde{e} + \tilde{e} &= e(\cdot, T), \quad \text{in } \Omega; \\ \tilde{e} &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{32}$$

where $\delta \in \mathbb{R}^+$ denotes the filter width. The weak formulation of the regularized problem (32) consists in find $\tilde{e} \in H_0^1(\Omega)$ such that

$$\delta^2(\nabla\tilde{e}, \nabla v) + (\tilde{e}, v) = (e(\cdot, T), v), \quad \forall v \in H_0^1(\Omega). \tag{33}$$

Thus, the δ-norm related to (33) is defined by

$$\|\tilde{e}\|_\delta^2 = \|\tilde{e}\|^2 + \|\delta\nabla\tilde{e}\|^2, \tag{34}$$

that can be expressed by

$$\|\tilde{e}\|_\delta^2 = (e, \tilde{e}), \tag{35}$$

by taking $v = \tilde{e}$ in (33). In order to associate the regularized error with the primal problem (4) we make use of the dual problem (26). Using (18), (19) and integrating by parts we get

$$\begin{aligned} R_h(\varphi) &= (\partial_t e, \varphi) + a(e, \varphi), \\ &= (e(\cdot, T), \varphi(\cdot, T)) - (e(\cdot, 0), \varphi(\cdot, 0)) + \Big[(e, -\partial_t\varphi) + a(e, \varphi)\Big], \quad \forall\varphi \in V,\ t \in I. \end{aligned} \tag{36}$$

By using (26) it follows that the functional $J(u - u_h) = (e(\cdot, T), \psi_T)$ may be written in the following way

$$J(e) = (e(\cdot, T), \psi_T) = R_h(\varphi) + (e(\cdot, 0), \varphi(\cdot, 0)). \tag{37}$$

If we choose $\psi_T = \frac{\tilde{e}}{\|\tilde{e}\|_\delta}$ in (37) and put $u_h(\cdot, 0) = u_0$ we obtain the expression

$$J(e) = \|\tilde{e}\|_\delta = (\partial_t e, \varphi) + a(e, \varphi). \tag{38}$$

In this case, our quantity of interest is defined by

$$J(u - u_h) = \|\tilde{u} - \tilde{u}_h\|_\delta. \tag{39}$$

Supposing that $a(\cdot, \cdot)$ in (25) represents an advection–diffusion operator, the stability of the dual problem (26) in the special case of regularized data is given in the next theorem, demonstrated in [1]. The multiscale decomposition assumption (7) of the velocities field plays an important role in the proof of this theorem.

Theorem 3.1 (*Stability of the Dual Problem*). *Let φ be the weak solution to (26), with $\psi_T \in H_0^1(\Omega)$ and Ω convex. Assume that the velocity field satisfies (7). Then*

$$\sup_{t\in I} \|\nabla\varphi(\cdot, t)\| + T^{-1}\|\nabla\varphi\|_Q + T^{-1}\|\partial_t\varphi\|_Q + |\mu^{1/2}\varphi|_{L^2(I;H^2(\Omega))} \preccurlyeq C_{\tau_F,T}\|\nabla\psi_T\|, \tag{40}$$

where

$$C_{\tau_F,T} = e^{c_\Lambda \frac{T}{\tau_F}}, \tag{41}$$

with τ_F given by (8) and c_Λ is a moderate constant.

Using the stability of the dual problem we obtain a superior bound to the stability factor $C_s = \|\varphi\|_{L^\infty(I;V)}$. This is achieved by choosing $\psi_T = \frac{\tilde{e}}{\|\tilde{e}\|_\delta}$ in (40), so that

$$C_s \preccurlyeq \delta^{-1} e^{c_\Lambda \frac{T}{\tau_F}}. \tag{42}$$

The quantity of interest (39) satisfies the following inequality

$$|J(u - u_h)| \leq C_s \|R_h\|_{L^1(I;V')}. \tag{43}$$

In the next two sections we will show how the weak residual term $\|R_h\|_{L^1(I;V')}$ can be bounded when a stabilized finite element method is used, herein we consider the CIP and SUPG methods. Consequently, a posteriori and a priori estimates are obtained for any quantity stable in the sense of Assumption 3.1 and in particular for the regularized error.

4. Continuous interior penalty finite element method

The Continuous Interior Penalty method (CIP) is a symmetric stabilization method proposed in [16] and analyzed further in [2]. This method consists in adding a weakly consistent, dissipative operator to the standard Galerkin formulation. In this work, we consider the version studied in [2] where the dissipative operator consists in a penalty on the jump of the gradient over element faces, given by

$$s_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \sum_{F \in \partial K \setminus \Omega} \langle h_F^2 \gamma \|\boldsymbol{\beta} \cdot \boldsymbol{\eta}_F\|_{L^\infty(F)} [\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!], [\![\nabla v_h \cdot \boldsymbol{\eta}_F]\!] \rangle_F, \tag{44}$$

where $[\![x]\!]$ denotes the jump of x over F and $\langle \cdot, \cdot \rangle_F$ the L^2-scalar product over F. Thus, the CIP stabilized finite element method is given by: for $t > 0$, find $u_h \in V_h$ such that $u_h(x, 0) = u_0(x)$ and

$$(\partial_t u_h, v_h) + a(u_h, v_h) + s_h(u_h, v_h) = \ell(v_h), \quad \forall v_h \in V_h. \tag{45}$$

Our aim is to show that the weak residual of (45) is bounded by a posteriori and a priori quantities on account of the presence of the operator $s_h(\cdot, \cdot)$ in the numerical formulation. The next three lemmas are important for the understanding of what follows, and they are proved in [1].

Lemma 4.1. *Let*

$$\bar{s}_h(u_h, v_h) = \sum_{K \in \mathcal{T}_h} \sum_{F \in \partial K \setminus \Omega} \langle h_F^2 \gamma \|\overline{\boldsymbol{\beta}} \cdot \boldsymbol{\eta}_F\|_{L^\infty(F)} [\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!], [\![\nabla v_h \cdot \boldsymbol{\eta}_F]\!] \rangle_F. \tag{46}$$

Then

$$s_h(u_h, v_h) \preccurlyeq \bar{s}_h(u_h, v_h) + h^{1/2} \|\mu^{1/2} \nabla u_h\|^2$$

and

$$\bar{s}_h(u_h, v_h) \preccurlyeq s_h(u_h, v_h) + h^{1/2} \|\mu^{1/2} \nabla u_h\|^2.$$

Proof. We refer the reader to Ref. [1]. ■

Lemma 4.2. *Assume that $\boldsymbol{\beta} \cdot \boldsymbol{\eta}_{\partial\Omega} = 0$ then there holds*

$$\inf_{v_h \in V_h} \|h^{1/2}(\overline{\boldsymbol{\beta}} \cdot \nabla u_h - v_h)\| \preccurlyeq \bar{s}_h(u_h, u_h)^{1/2} + h^{1/2} \|\overline{\boldsymbol{\beta}}\|_{W^{1,\infty}(\Omega)} \|u_h\|,$$

where $\bar{s}_h(u_h, u_h)$ is given by (46).

Proof. We refer the reader to Ref. [1]. ■

Lemma 4.3 (*Stability of the CIP Method*). *Let u_h be the solution of (45), with $\gamma > 0$, and consider the norm associated to the CIP method*

$$|\!|\!| u_h |\!|\!|_{cip}^2 = \int_I \Big(\|\mu^{1/2} \nabla u_h\|^2 + s_h(u_h, u_h) \Big) \, \mathrm{d}t, \tag{47}$$

then there holds

$$\sup_{t \in I} \|u_h(t)\| + |\!|\!| u_h |\!|\!|_{cip} \preccurlyeq \int_I \|f\| \, \mathrm{d}t + \|u_0\|. \tag{48}$$

Proof. We refer the reader to Ref. [1]. ■

4.1. Error representation—CIP method

By subtracting side to side (45) from (16) and defining the numerical error $e = u - u_h \in V$, we get the equation

$$(\partial_t e, v_h) + a(e, v_h) - s_h(u_h, v_h) = 0, \quad \forall v_h \in V_h. \tag{49}$$

On the other hand,

$$\begin{aligned}(\partial_t e, v - \pi_h v) + a(e, v - \pi_h v) &= (\partial_t e, v) + a(e, v) - \Big[(\partial_t e, \pi_h v) + a(e, \pi_h v)\Big], \\ &= (\partial_t e, v) + a(e, v) - s_h(u_h, \pi_h v), \quad \forall v \in V.\end{aligned} \tag{50}$$

From (50) and for each $t \in I$ the weak residual R_h^{cip} associated with (45) is given by

$$\begin{aligned}R_h^{cip}(v) &= (\partial_t e, v) + a(e, v) \\ &= \ell(v - \pi_h v) - \Big[(\partial_t u_h, v - \pi_h v) + a(u_h, v - \pi_h v)\Big] + s_h(u_h, \pi_h v), \quad \forall v \in V.\end{aligned} \tag{51}$$

The next theorem shows that the residual $R_h^{cip}(\cdot)$ can be controlled by an a posteriori quantity.

Theorem 4.1. *Let R_h^{cip} be defined by (51), with u and u_h solutions of (4) and (45) respectively. Then*

$$\begin{aligned}\|R_h^{cip}\|_{L^1(I,V')} \preccurlyeq h^{1/2} \int_I \Big(&\inf_{w_h \in V_h} \|h^{1/2}(f + \mu\Delta u_h + w_h)\|_{\mathcal{T}} + \inf_{w_h \in V_h} \|h^{1/2}(\boldsymbol{\beta} \cdot \nabla u_h - w_h)\| \\ &+ s_h(u_h, u_h)^{1/2} + \|\mu[\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_{\mathcal{F}}\Big)\,\mathrm{d}t.\end{aligned} \tag{52}$$

Proof. For this demonstration, we start from (51) and use the Cauchy–Schwarz inequality, (13) and (14). The diffusive part of the bilinear form $a(\cdot,\cdot)$ is limited by integrating it by parts on each element K. Firstly, taking v such that $\sup_{t\in I} \|v\|_V = 1$ we notice that

$$\begin{aligned}(f, v - \pi_h v) - (\mu \nabla u_h, \nabla(v - \pi_h v)) &= \int_\Omega (f + \mu\Delta u_h)(v - \pi_h v)\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu(\nabla u_h \cdot \boldsymbol{\eta}_{\partial K})(v - \pi_h v)\,\mathrm{d}s \\ &= \int_\Omega (f + \mu\Delta u_h + w_h)(v - \pi_h v)\,\mathrm{d}\mathbf{x} - \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu(\nabla u_h \cdot \boldsymbol{\eta}_{\partial K})(v - \pi_h v)\,\mathrm{d}s, \\ &\le \Big(\inf_{w_h\in V_h} \|h^{1/2}(f + \mu\Delta u_h + w_h)\|_{\mathcal{T}}\Big)\|h^{-1/2}(v - \pi_h v)\| \\ &\quad + \|\mu[\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_{\mathcal{F}} \|v - \pi_h v\|_{\mathcal{F}}.\end{aligned} \tag{53}$$

Then it follows that, since by (12) and $\|v\|_V = 1$, $\|h^{-1/2}(v - \pi_h v)\| + \|v - \pi_h v\|_{\mathcal{F}} \preccurlyeq h^{\frac{1}{2}}$,

$$\begin{aligned}(f, v - \pi_h v)_Q - (\mu\nabla u_h, \nabla(v - \pi_h v))_Q &\preccurlyeq \int_I \Big(\inf_{w_h\in V_h} \|h^{1/2}(f + \mu\Delta u_h + w_h)\|_{\mathcal{T}} + \|\mu[\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_{\mathcal{F}}\Big)\,\mathrm{d}t \\ &\quad \times \sup_{t\in I}\Big(\|h^{-1/2}(v - \pi_h v)\| + \|v - \pi_h v\|_{\mathcal{F}}\Big) \\ &\preccurlyeq h^{1/2} \int_I \Big(\inf_{w_h\in V_h} \|h^{1/2}(f + \mu\Delta u_h + w_h)\|_{\mathcal{T}} + \|\mu[\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_{\mathcal{F}}\Big)\,\mathrm{d}t.\end{aligned} \tag{54}$$

The first order part of the form $a(\cdot,\cdot)$ is given by

$$\begin{aligned}(\partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h, v - \pi_h v)_Q &= (\partial_t u_h, v - \pi_h v)_Q + (\boldsymbol{\beta}\cdot\nabla u_h, v - \pi_h v)_Q \\ &= (\boldsymbol{\beta}\cdot\nabla u_h - w_h, v - \pi_h v)_Q, \quad w_h \in V_h,\end{aligned} \tag{55}$$

where we have used the fact of $\partial_t u_h \in V_h$ and the orthogonality of the L^2-projection. Therefore

$$(\partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h, v - \pi_h v)_Q \preccurlyeq h^{1/2}\Big(\int_I \inf_{w_h\in V_h} \|h^{1/2}(\boldsymbol{\beta}\cdot\nabla u_h - w_h)\|\,\mathrm{d}t\Big) \sup_{t\in I} \|v\|_V. \tag{56}$$

Finally, from the symmetry and positive semi-definite properties of the operator $s_h(\cdot,\cdot)$ and using the Schwarz inequality we get

$$|s_h(u_h, \pi_h v)| \le s_h(u_h, u_h)^{1/2} s_h(\pi_h v, \pi_h v)^{1/2}.$$

By using the definition of the operator $s_h(\cdot,\cdot)$, the inverse inequality (10) and the stability of the operator π_h, we have that

$$s_h(\pi_h v, \pi_h v) \preccurlyeq h\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}. \tag{57}$$

Then,

$$\begin{aligned}\int_I s_h(u_h, \pi_h v)\,\mathrm{d}t &\preccurlyeq h^{1/2}\|\boldsymbol{\beta}\|^{1/2}_{L^\infty(Q)}\Big(\int_I s_h(u_h,u_h)^{1/2}\,\mathrm{d}t\Big)\\ &\preccurlyeq h^{1/2}\Big(\int_I s_h(u_h,u_h)^{1/2}\,\mathrm{d}t\Big).\end{aligned} \tag{58}$$

Collecting the upper bounds (54)–(58) and using the definition of the norms (22)–(23) and of the residual (51) we obtain the desired result. ■

An a priori bound of the weak residual $R_h^{cip}(\cdot)$ is obtained with the help of Lemmas 4.1, 4.2 and Theorem 4.1. The next theorem shows how this can be done.

Theorem 4.2. *Let R_h^{cip} be defined by (51), with u and u_h solutions of (4) and (45) respectively. Assume that $Pe_h > 1$, then there holds*

$$\|R_h^{cip}\|_{L^1(I,V')} \preccurlyeq h^{1/2}(1+h^{1/2})\Big(\int_I \|f\|\;\mathrm{d}t + \|u_0\|\Big). \tag{59}$$

Proof. The result follows from Theorem 4.1 by bounding all the residual terms and using the stability results of the CIP stabilized method, given by (48). The first term on the right hand side of (52) is bounded as follows,

$$\begin{aligned}\int_I \inf_{w_h\in V_h}\|h^{1/2}(f+\mu\Delta u_h + w_h)\|_{\mathcal{T}}\,\mathrm{d}t &\preccurlyeq h^{1/2}\int_I \|f\|\,\mathrm{d}t + h^{-1/2}\mu^{1/2}\Big(\int_I \|\mu^{1/2}\nabla u_h\|^2\,\mathrm{d}t\Big)^{1/2}\\ &\preccurlyeq h^{1/2}\int_I \|f\|\,\mathrm{d}t + |\!|\!| u_h |\!|\!|_{cip}\,.\end{aligned}$$

Using a Cauchy–Schwarz inequality in time, the term associated to contributions on the faces is limited in the following way:

$$\begin{aligned}\int_I \|\mu[\![\nabla u_h\cdot\boldsymbol{\eta}_F]\!]\|_{\mathcal{F}}\,\mathrm{d}t &\le \Big(\int_I \mathrm{d}t\Big)^{1/2}\Big(\int_I \|\mu[\![\nabla u_h\cdot\boldsymbol{\eta}_F]\!]\|^2_{\mathcal{F}}\,\mathrm{d}t\Big)^{1/2}\\ &\preccurlyeq \mu^{1/2}\Big(\int_I \|\mu^{1/2}[\![\nabla u_h\cdot\boldsymbol{\eta}_F]\!]\|^2_{\mathcal{F}}\,\mathrm{d}t\Big)^{1/2}\\ &\preccurlyeq \|\boldsymbol{\beta}\|^{1/2}_{L^\infty(\Omega)}h^{1/2}\Big(\int_I \|\mu^{1/2}[\![\nabla u_h\cdot\boldsymbol{\eta}_F]\!]\|^2_{\mathcal{F}}\,\mathrm{d}t\Big)^{1/2}\\ &\preccurlyeq \|\boldsymbol{\beta}\|^{1/2}_{L^\infty(\Omega)}\Big(\int_I \|\mu^{1/2}\nabla u_h\|^2\,\mathrm{d}t\Big)^{1/2}\\ &\preccurlyeq |\!|\!| u_h |\!|\!|_{cip}\,,\end{aligned}$$

where we have used the fact that $\mu < \|\boldsymbol{\beta}\|_{L^\infty(\Omega)}h$ and the inverse inequality (9). Resorting to the Cauchy–Schwarz inequality in time the stabilization term can be bounded as follows

$$\begin{aligned}\int_I s_h(u_h,u_h)^{1/2}dt &\le \Big(\int_I \mathrm{d}t\Big)^{1/2}\Big(\int_I s_h(u_h,u_h)\,\mathrm{d}t\Big)^{1/2}\\ &\preccurlyeq \Big(\int_I s_h(u_h,u_h)\,\mathrm{d}t\Big)^{1/2}\\ &\preccurlyeq |\!|\!| u_h |\!|\!|_{cip}\,.\end{aligned}$$

Finally, to limit the second term on the right hand side of (52) the stabilization method plays an important role. First, we use the velocity decomposition assumption (7) and the Cauchy–Schwarz inequality in time in order to obtain

$$\int_I \inf_{w_h\in V_h}\|h^{1/2}(\boldsymbol{\beta}\cdot\nabla u_h - w_h)\|\,\mathrm{d}t \preccurlyeq \Big(\int_I \inf_{w_h\in V_h}\|h^{1/2}(\overline{\boldsymbol{\beta}}\cdot\nabla u_h - w_h)\|^2\,\mathrm{d}t\Big)^{\frac{1}{2}} + \|h^{1/2}\boldsymbol{\beta}'\cdot\nabla u_h\|_Q\,.$$

By Lemmas 4.2 and 4.1 we have that

$$
\begin{aligned}
\left(\int_I \inf_{w_h\in V_h} \|h^{1/2}(\overline{\boldsymbol\beta}\cdot\nabla u_h - w_h)\|^2\,\mathrm{d}t\right)^{\frac12} &\preccurlyeq h^{1/2}\|\overline{\boldsymbol\beta}\|_{W^{1,\infty}(\Omega)}\|u_h\|_Q + \left(\int_I \bar s_h(u_h,u_h)\,\mathrm{d}t\right)^{1/2}\\
&\preccurlyeq h^{1/2}\|\overline{\boldsymbol\beta}\|_{W^{1,\infty}(\Omega)}\sup_{t\in I}\|u_h\| + \left(\int_I s_h(u_h,u_h)\,\mathrm{d}t\right)^{1/2}\\
&\preccurlyeq \max\{h^{1/2}\|\overline{\boldsymbol\beta}\|_{W^{1,\infty}(\Omega)}, \|\boldsymbol\beta\|^{1/2}_{L^\infty(\Omega)}\}\left(\sup_{t\in I}\|u_h(t)\| + |||u_h|||_{cip}\right).
\end{aligned}
$$

On the other hand, using the assumption $\|\boldsymbol\beta'\|^2_{L^\infty(Q)} \preccurlyeq \mu$, we have

$$
\begin{aligned}
\|h^{1/2}\boldsymbol\beta'\cdot\nabla u_h\|_Q &\le h^{1/2}\|\boldsymbol\beta'\|_{L^\infty(Q)}\|\nabla u_h\|_Q\\
&\preccurlyeq h^{1/2}\|\mu^{1/2}\nabla u_h\|_Q\\
&\le h^{1/2}\,|||u_h|||_{cip}\,.
\end{aligned}
$$

By collecting terms and applying Lemma 4.3 we obtain the desired result. ■

Remark 4.1. To obtain an a priori bound on $\|R_h^{cip}\|_{L^1(I;V')}$ independent of the diffusivity coefficient μ (see Eq. (59)), for the case $Pe_h > 1$, the stabilized method plays an important role, especially to bound the second term (advective term) of (52). This term cannot be bounded by using only the standard Galerkin method.

A framework to obtain a posteriori and a priori estimates for the output $J(\cdot)$ in the CIP method context is given by the next theorem.

Theorem 4.3. *Let J be a functional that represents some quantity of interest related to the problem (4). We assume that (4) has the continuous dependence property (24). If u and u_h are the solutions of (4) and (45), respectively, then $J(\cdot)$ satisfies*

(i) $$|J(u-u_h)| \le \Theta(\omega(u_h)h^{1/2}), \tag{60}$$

where $\omega(u_h)$ is a posteriori quantity given by

$$\omega(u_h) = \int_I\left(\inf_{w_h\in V_h}\|h^{1/2}(f+\mu\Delta u_h + w_h)\|_{\mathcal T} + \inf_{w_h\in V_h}\|h^{1/2}(\boldsymbol\beta\cdot\nabla u_h - w_h)\| + s_h(u_h,u_h)^{1/2} + \|\mu[\![\nabla u_h\cdot\boldsymbol\eta_F]\!]\|_{\mathcal F}\right)\mathrm{d}t, \tag{61}$$

(ii) *and*

$$|J(u-u_h)| \le \Theta(C_{f,T,u_0}\,h^{1/2}), \tag{62}$$

where

$$C_{f,T,u_0} = (1+h^{1/2})\left(\int_I \|f\|\,\mathrm{d}t + \|u_0\|\right). \tag{63}$$

Proof. Let $e = u - u_h \in V$. Eq. (51) implies that the error e satisfies Eq. (4) with right hand side $R_h^{cip}(\varphi)$, that is,

$$(\partial_t e,\varphi) + a(e,\varphi) = R_h^{cip}(\varphi).$$

By Theorems 4.1 we have $\|R_h^{cip}\|_{L^1(I;V')} \le \omega(u_h)h^{1/2}$. The continuous dependence assumption allows us to conclude the claimed result (60). Analogously, we use Theorem 4.2 to obtain the bound $\|R_h^{cip}\|_{L^1(I;V')} \le C_{f,T,u_0}\,h^{1/2}$ so that the result (62) follows immediately from (24). ■

This theorem can be used to obtain robust a posteriori and a priori error estimates for quantities of interest, regardless of the Péclet number, for the transient advection–diffusion equations. The next corollary shows an example of how this can be done by considering the regularized error discussed in Section 3.1.

Corollary 4.1 (*A Posteriori and a Priori Error Estimates for the Regularized Error*)**.** *Let $\tilde e$ be defined by (33) and assume that (4) has the continuous dependence property (24). If u and u_h are the solutions of (4) and (45), respectively, then*

(i) *(a posteriori estimate)*

$$\|\tilde e\|_\delta \preccurlyeq \omega(u_h)e^{c_\Lambda T/\tau_F}\left(\frac{h}{\delta^2}\right)^{1/2}, \tag{64}$$

where $\omega(u_h)$ is a posteriori quantity given by (61).

(ii) *(a priori estimate)*

$$\|\tilde{e}\|_\delta \preccurlyeq C_{f,T,u_0}\, e^{c_\Lambda T/\tau_F} \left(\frac{h}{\delta^2}\right)^{1/2}, \tag{65}$$

where C_{f,T,u_0} *is a priori bound given by* (63).

Proof. By taking $J(u - u_h) = \|\tilde{e}\|_\delta$, follows from Theorem 4.3, item (i), that

$$\|\tilde{e}\|_\delta \preccurlyeq \Theta(\omega(u_h)h^{1/2}).$$

The a posterior estimate (64) is obtained by noting that for this choice of functional, $\Theta(\omega(u_h)h^{1/2}) = C_s\omega(u_h)h^{1/2}$, as explained in Section 3.1, Eqs. (39)–(43). The constant C_s is given in Eq. (42).

The a priori error estimate (65) is a consequence of Theorem 4.3, item (ii). ■

The expression (65) shows that the a priori estimate for the regularized error is independent both of the Sobolev norms of the exact solution and of the Péclet number, but depends on L^2-norm of data and the exponential factor (41).

5. SUPG finite element space semi-discretization

One of the most known numerical methodologies to solve convection-dominated transport problem is the Streamline Upwind Petrov–Galerkin method (SUPG), or Streamline Diffusion method, introduced in [3] and analyzed in [4]. This method consists in finding $u_h \in V_h, \forall t \in I$, such that

$$(\partial_t u_h, v_h) + a(u_h, v_h) + \sum_{K\in\mathcal{T}_h} \int_K (\mathcal{L}u_h - f)\delta_K \boldsymbol{\beta}\cdot\nabla v_h\, \mathrm{d}\mathbf{x} = (f, v_h), \quad \forall v_h \in V_h, \tag{66}$$

where

$$\mathcal{L}u_h = \partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h - \mu\Delta u_h$$

and δ_K denotes the stabilization parameter, defined by

$$\delta_K = \begin{cases} \dfrac{h}{\|\boldsymbol{\beta}\|_{L^\infty(\Omega)}}\lambda, & \text{if } Pe_h > 1;\\ 0, & \text{if } Pe_h \le 1,\end{cases} \tag{67}$$

where $Pe_h = \frac{\|\boldsymbol{\beta}\|_{L^\infty(\Omega)} h}{\mu}$ is the mesh Péclet number and $\lambda \in \mathbb{R}^+$ is a coefficient to be properly chosen. We assume that $Pe_h > 1$ so that $\mu < \|\boldsymbol{\beta}\|_{L^\infty(\Omega)} h$.

The general behavior of the solution of stationary problems using the SUPG method is well understood whereas for the transient problems the situation is less clear. In [17] the stability and convergence for the SUPG space semi-discretization of the transient convection–diffusion equation was shown. In this work we use the stability result of [17] given by Theorem 5.1 to obtain a priori bound on the weak residual of the SUPG method. We assume that $\boldsymbol{\beta}(\boldsymbol{x}, t) = \boldsymbol{\beta}(\boldsymbol{x})$.

Theorem 5.1 (*Stability of the SUPG Method*). *Let* u_h *be the solution to* (66), *with* $0 < \lambda < \frac{1}{4}\min\left\{\frac{1}{c_i}, \frac{1}{c_i^2}\right\}$, *where* c_i *is the constant in the inverse inequality* (9). *We assume that* $\int_I \|\partial_t f\|^2\,\mathrm{d}t$ *is bounded. Then,*

$$\sup_{t\in I} \|u_h\|_\beta^2 + |||u_h|||_{supg}^2 \preccurlyeq \int_I \|f - \delta_K \partial_t f\|^2\,\mathrm{d}t + \sup_{t\in I}\|\delta_K f\|^2 + \|u_0\|_\beta^2, \tag{68}$$

where the norms $\|u_h\|_\beta$ *and* $|||u_h|||_{supg}$ *are defined by*

$$\|u_h\|_\beta^2 = \|u_h\|^2 + \|\delta_K \boldsymbol{\beta}\cdot\nabla u_h\|^2 \tag{69}$$

and

$$|||u_h|||_{supg}^2 = \int_I \left(\|\delta_K^{1/2}(\partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h)\|^2 + \|\mu^{1/2}\nabla u_h\|^2\right)\mathrm{d}t. \tag{70}$$

5.1. Error representation—SUPG method

The formulation (66) is strongly consistent. Indeed,

$$(\partial_t e, v_h) + a(e, v_h) + \sum_{K\in\mathcal{T}_h} \int_K (\mathcal{L}e)\delta_K \boldsymbol{\beta}\cdot\nabla v_h \,\mathrm{d}\mathbf{x} = 0, \quad \forall v_h \in V_h. \tag{71}$$

This means that

$$(\partial_t e, v_h) + a(e, v_h) = \sum_{K\in\mathcal{T}_h} \int_K (\mathcal{L}u_h - f)\delta_K \boldsymbol{\beta}\cdot\nabla v_h \,\mathrm{d}\mathbf{x}, \quad \forall v_h \in V_h. \tag{72}$$

By using (72) and taking $v \in V$, we have that

$$\begin{aligned}(\partial_t e, v - \pi_{\tilde{K}} v) + a(e, v - \pi_{\tilde{K}} v) &= (\partial_t e, v) + a(e, v) - \Big[(\partial_t e, \pi_{\tilde{K}} v) + a(e, \pi_{\tilde{K}} v)\Big] \\ &= (\partial_t e, v) + a(e, v) - \sum_{K\in\mathcal{T}_h} \int_K (\mathcal{L}u_h - f)\delta_K \boldsymbol{\beta}\cdot\nabla\pi_{\tilde{K}} v \,\mathrm{d}\mathbf{x}. \end{aligned} \tag{73}$$

From (73) the weak residual R_h^{supg} associated with (66) is given by

$$R_h^{supg}(v) = \ell(v - \pi_{\tilde{K}} v) - \Big[(\partial_t u_h, v - \pi_{\tilde{K}} v) + a(u_h, v - \pi_{\tilde{K}} v)\Big] + \sum_{K\in\mathcal{T}_h} \int_K (\mathcal{L}u_h - f)\delta_K \boldsymbol{\beta}\cdot\nabla\pi_{\tilde{K}} v \,\mathrm{d}\mathbf{x}, \quad \forall t \in I. \tag{74}$$

In the next theorem we obtain an a posteriori quantity in order to control the weak norm of the residual $R_h^{supg}(\cdot)$.

Theorem 5.2. *Let R_h^{supg} be defined by (74) with u and u_h solutions of (4) and (66), respectively, and suppose that $0 < \lambda \leq 1$. Then,*

$$\|R_h^{supg}\|_{L^1(I,V')} \preccurlyeq h^{1/2} \int_I \Big[\|h^{1/2} r_h\|_{\mathcal{T}} + \|\mu [\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_{\mathcal{F}}\Big] \,\mathrm{d}t, \tag{75}$$

where

$$r_h = \mathcal{L}u_h - f = \partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h - \mu\Delta u_h - f. \tag{76}$$

Proof. First, we note that taking v satisfying $\sup_{t\in I}\|v\|_V = 1$, Eq. (74) can be written

$$\begin{aligned} R_h^{supg}(v) &= (f, v - \pi_{\tilde{K}} v) - \Big[(\partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h, v - \pi_{\tilde{K}} v) + (\mu\nabla u_h, \nabla(v - \pi_{\tilde{K}} v))\Big] \\ &\quad + \sum_{K\in\mathcal{T}_h} \int_K (\mathcal{L}u_h - f)\delta_K \boldsymbol{\beta}\cdot\nabla\pi_{\tilde{K}} v d\Omega, \quad \forall t \in I; \\ &= -\sum_{K\in\mathcal{T}_h} \int_K r_h (v - \pi_{\tilde{K}} v) d\Omega - \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\mu\nabla u_h \cdot \boldsymbol{\eta}_{\partial K})(v - \pi_{\tilde{K}} v) d\Gamma \\ &\quad + \sum_{K\in\mathcal{T}_h} \int_K r_h \delta_K \boldsymbol{\beta}\cdot\nabla\pi_{\tilde{K}} v d\Omega, \quad \forall t \in I. \end{aligned} \tag{77}$$

Now we integrate (77) in time and use the Cauchy–Schwarz inequality, (13) and (14). The first term of the right hand side of (77) is bounded as follows

$$\begin{aligned} \int_I \sum_{K\in\mathcal{T}_h} \int_K r_h (v - \pi_{\tilde{K}} v) \,\mathrm{d}\mathbf{x}\,\mathrm{d}t &\preccurlyeq h^{1/2} \int_I \sum_{K\in\mathcal{T}_h} \|h^{1/2} r_h\|_K \|\nabla v\|_{\tilde{K}} \,\mathrm{d}t \\ &\preccurlyeq h^{1/2} \Big[\int_I \Big(\sum_{K\in\mathcal{T}_h} \|h^{1/2} r_h\|_K^2\Big)^{1/2} \,\mathrm{d}t\Big]. \end{aligned} \tag{78}$$

The second term is bounded according to

$$\begin{aligned} \int_I \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\mu\nabla u_h \cdot \boldsymbol{\eta}_{\partial K})(v - \pi_{\tilde{K}} v) \,\mathrm{d}s\,\mathrm{d}t &\preccurlyeq h^{1/2} \int_I \sum_{F\in\mathcal{F}} \|\mu [\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_F \|\nabla v\|_{\tilde{K}} \,\mathrm{d}t \\ &\preccurlyeq h^{1/2} \Big[\int_I \Big(\sum_{F\in\mathcal{F}} \|\mu [\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_F^2\Big)^{1/2} \,\mathrm{d}t\Big]. \end{aligned} \tag{79}$$

If $\lambda \le 1$ then $\delta_K \|\boldsymbol{\beta}\|_{L^\infty(\Omega)} \le h$. Thus, for the last term of (77) we have

$$\int_I \sum_{K\in\mathcal{T}_h} \int_K r_h \delta_K \boldsymbol{\beta}\cdot\nabla\pi_{\tilde{K}} v \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \le \int_I \sum_{K\in\mathcal{T}_h} \delta_K \|r_h\|_K \|\boldsymbol{\beta}\|_{L^\infty(\Omega)} \|\nabla\pi_{\tilde{K}} v\|_K \,\mathrm{d}t$$
$$\lesssim h^{1/2}\Big[\int_I \Big(\sum_{K\in\mathcal{T}_h} \|h^{1/2} r_h\|_K^2\Big)^{1/2} \mathrm{d}t\Big]. \tag{80}$$

The desired result is obtained by collecting the upper bounds (78)–(80) and using (22), (23) and (74). ■

The stability result given in Theorem 5.1 is used now to obtain an a priori bound of the weak norm of $R_h^{supg}(\cdot)$, as follows.

Theorem 5.3. *Let R_h^{supg} be defined by (74) with u and u_h solutions of (4) and (66), respectively, and suppose that $0 < \lambda \le \min\left\{\frac{1}{4c_i^2}, \frac{1}{4c_i}, 1\right\}$, with c_i the constant in Eq. (9). Then there holds*

$$\|R_h^{supg}\|_{L^1(I,V')}\,\mathrm{d}t \lesssim h^{1/2} T\Big[h^{1/2}\sup_{t\in I}\|f\| + C_\lambda T^{1/2}\Big(\int_I \|f-\delta_K\partial_t f\|^2\,\mathrm{d}t + \sup_{t\in I}\|\delta_K f\|^2 + \|u_0\|_\beta^2\Big)^{1/2}\Big], \tag{81}$$

where the constant C_λ is given by

$$C_\lambda = \|\boldsymbol{\beta}\|_{L^\infty(\Omega)}^{1/2}\left(2 + \frac{1}{\sqrt{\lambda}}\right).$$

Proof. To prove this theorem we use Cauchy–Schwarz inequality in time in Eq. (75) and the stability results of Theorem 5.1. Using (77) and proceeding in the same way as in the proof of Theorem 5.2 we can write

$$\|R_h^{supg}\|_{L^1(I,V')} \lesssim h^{1/2}\int_I \Big[\|h^{1/2} r_h\|_{\mathcal{T}} + \|\mu[\![\nabla u_h\cdot\boldsymbol{\eta}_F]\!]\|_{\mathcal{F}}\Big]\mathrm{d}t$$
$$\lesssim h^{1/2}\int_I \Big(\|h^{1/2} f\| + \|h^{1/2}(\partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h)\| + \|h^{1/2}\mu\Delta u_h\|_{\mathcal{T}}\Big)\mathrm{d}t + h^{1/2}\int_I \|\mu[\![\nabla u_h\cdot\boldsymbol{\eta}_F]\!]\|_{\mathcal{F}}\,\mathrm{d}t. \tag{82}$$

Now, we can evaluate each term on the right hand side of (82) separately. For the first one, we have

$$\int_I \|h^{1/2} f\|\,\mathrm{d}t \le h^{1/2} T \sup_{t\in I}\|f\|.$$

The second term on the right hand side of (82) results in

$$\int_I \|h^{1/2}(\partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h)\|\,\mathrm{d}t \le \Big(\int_I \mathrm{d}t\Big)^{1/2}\Big(\int_I \|h^{1/2}(\partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h)\|^2\,\mathrm{d}t\Big)^{1/2}$$
$$\le T^{1/2} h^{1/2}\delta_K^{-1/2}\Big(\int_I \|\delta_K^{1/2}(\partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h)\|^2\,\mathrm{d}t\Big)^{1/2}$$
$$\le h^{1/2} T^{1/2}\delta_K^{-1/2}\,|\!|\!| u_h |\!|\!|_{supg}.$$

Using the inverse inequality (11) we have

$$h^{1/2}\|\mu\Delta u_h\|_K \lesssim h^{-1/2}\mu^{1/2}\|\mu^{1/2}\nabla u_h\|_K.$$

Then, we can bound the term associated to the diffusion operator as follows,

$$\int_I \|h^{1/2}\mu\Delta u_h\|_{\mathcal{T}}\,\mathrm{d}t \lesssim \int_I h^{-1/2}\mu^{1/2}\|\mu^{1/2}\nabla u_h\|\,\mathrm{d}t$$
$$\lesssim h^{-1/2}\mu^{1/2}\Big(\int_I \mathrm{d}t\Big)^{1/2}\Big(\int_I \|\mu^{1/2}\nabla u_h\|^2\,\mathrm{d}t\Big)^{1/2}$$
$$\le h^{-1/2} T^{1/2}\mu^{1/2}\,|\!|\!| u_h |\!|\!|_{supg}.$$

Finally, we have the bound on the term associated to contributions on the faces, that is given by

$$\begin{aligned}
\int_I \|\mu[\![\nabla u_h \cdot \boldsymbol{\eta}]\!]\|_{\mathcal{F}}\, \mathrm{d}t &\le \left(\int_I \mathrm{d}t\right)^{1/2}\left(\int_I \sum_{F\in\mathcal{F}} \|\mu[\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_F^2\, \mathrm{d}t\right)^{1/2} \\
&\le T^{1/2}\mu^{1/2}\left(\int_I \sum_{F\in\mathcal{F}} \|\mu^{1/2}[\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_F^2\, \mathrm{d}t\right)^{1/2} \\
&= T^{1/2}\mu^{1/2}h^{-1/2}h^{1/2}\left(\int_I \sum_{F\in\mathcal{F}} \|\mu^{1/2}[\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_F^2\, \mathrm{d}t\right)^{1/2} \\
&\preccurlyeq T^{1/2}\mu^{1/2}h^{-1/2}\left(\int_I \|\mu^{1/2}\nabla u_h\|^2\, \mathrm{d}t\right)^{1/2} \\
&\le h^{-1/2}T^{1/2}\mu^{1/2}\, |\!|\!| u_h |\!|\!|_{supg}\, .
\end{aligned}$$

By collecting those results, using Theorem 5.1 and the fact that $\mu^{1/2}h^{-1/2} < \|\boldsymbol{\beta}\|^{1/2}_{L^\infty(\Omega)}$ and $h^{1/2}\delta_K^{-1/2} = \frac{\|\boldsymbol{\beta}\|^{1/2}_{L^\infty(\Omega)}}{\sqrt{\lambda}}$ we obtain the desired result. ■

The same framework described for CIP method in order to obtain a posteriori and a priori estimates for quantities of interest is presented here for SUPG method. This result is given by the next theorem.

Theorem 5.4. *Let J be a functional that represents some quantity of interest related to the problem (4). We also assume that (4) has the continuous dependence property (24). If u and u_h are the solutions of (4) and (66), respectively, then $J(\cdot)$ satisfies*

(i)

$$|J(u-u_h)| \le \Theta(\omega(u_h)h^{1/2}), \tag{83}$$

where $\omega(u_h)$ is a posteriori quantity given by

$$\omega(u_h) = \int_I \left[\|h^{1/2} r_h\|_{\mathcal{T}} + \|\mu[\![\nabla u_h \cdot \boldsymbol{\eta}_F]\!]\|_{\mathcal{F}}\right] \mathrm{d}t, \tag{84}$$

with

$$r_h = \partial_t u_h + \boldsymbol{\beta}\cdot\nabla u_h - \mu\Delta u_h - f$$

(ii) *and*

$$|J(u-u_h)| \le \Theta(C_{f,T,u_0}\, h^{1/2}), \tag{85}$$

where

$$C_{f,T,u_0} = T\left[h^{1/2}\sup_{t\in I}\|f\| + C_\lambda T^{1/2}\left(\int_I \|f-\delta_K\partial_t f\|^2\, \mathrm{d}t + \sup_{t\in I}\|\delta_K f\|^2 + \|u_0\|_\beta^2\right)^{1/2}\right], \tag{86}$$

with

$$C_\lambda = \|\boldsymbol{\beta}\|^{1/2}_{L^\infty(\Omega)}\left(2+\frac{1}{\sqrt{\lambda}}\right).$$

Proof. The proof of this theorem is the same as that of Theorem 5.4. We just must replace $\|R_h^{cip}\|_{L^1(I;V')}$ by $\|R_h^{supg}\|_{L^1(I;V')}$ and its correspondents bounds. The a posteriori quantity bound for $\|R_h^{supg}\|_{L^1(I;V')}$ is given by Theorem 5.2 whilst the a priori bound by Theorem 5.3. ■

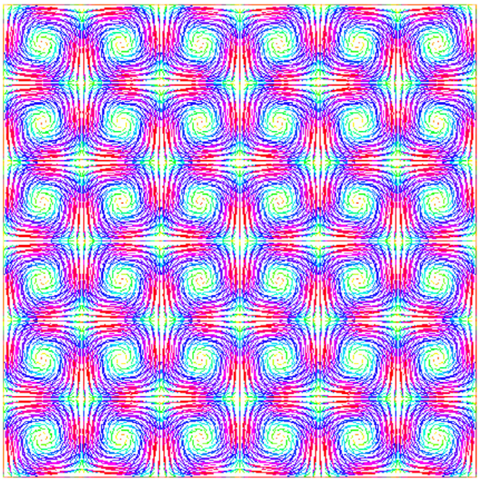
The following corollary shows an application of this theorem when the output represents the regularized error discussed in Section 3.1.

Corollary 5.1 (*A Posteriori and a Priori Error Estimates for the Regularized Error*)**.** *Let $\tilde{e}$ be defined by (33) and assume that (4) has the continuous dependence property (24). If u and u_h are the solutions of (4) and (66), respectively, then*

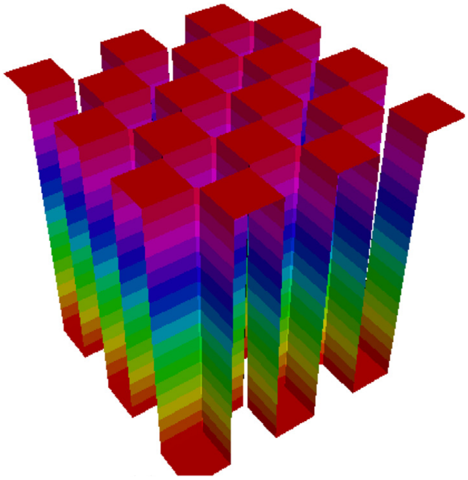
(i) *(a posteriori estimate)*

$$\|\tilde{e}\|_\delta \preccurlyeq \omega(u_h)e^{c_A T/\tau_F}\left(\frac{h}{\delta^2}\right)^{1/2}, \tag{87}$$

where $\omega(u_h)$ is a posteriori quantity given by (84).



(a) Velocity field.



(b) Initial solution.

Fig. 1. Velocities field and initial solution, using $k = 3$.

(ii) *(a priori estimate)*

$$\|\tilde{e}\|_\delta \preccurlyeq C_{f,T,u_0}\, e^{c_\Lambda T/\tau_F} \left(\frac{h}{\delta^2}\right)^{1/2}, \tag{88}$$

where C_{f,T,u_0} is a priori bound given by (86).

Proof. Similar to the proof of Corollary 4.1. ■

Here, again the a priori estimate for the regularized error only depends on L^2-norm of data and the exponential factor (41). In the next section the theoretical results are illustrated by computations on a scalar transport equation with no diffusion term, rough data and strongly varying velocity field.

6. Numerical experiments

In this section we consider the pure advection problem given by

$$\partial_t u + \boldsymbol{\beta} \cdot \nabla u = 0, \quad \text{in } \Omega \times (0, T); \tag{89}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \text{in } \Omega, \tag{90}$$

where $\Omega = [0, 1] \times [0, 1]$, $u_0(\boldsymbol{x})$ is the initial solution given by the *checkerboard* function, (see Fig. 1(b)), and $\boldsymbol{\beta} = (\beta_x, \beta_y)^t$, with

$$\beta_x = 2k\pi \sin(2k\pi x)\cos(2k\pi y); \tag{91}$$

$$\beta_y = -2k\pi \cos(2k\pi x)\sin(2k\pi y), \tag{92}$$

$k = 1, 2, 3, \ldots$, is the velocity field. It is straightforward to verify that $\nabla \cdot \boldsymbol{\beta} = 0$ and that $\boldsymbol{\beta}$ is a stationary solution to the incompressible Euler equations. This is a transport problem with infinite Péclet number (no diffusion term), rough data and strongly varying velocity field. Fig. 1(a) shows the velocity fields for $k = 3$. We evaluate the space convergence rates and the growth of the error in time for the quantity $J(e) = \|\tilde{e}\|_\delta$ for both CIP and SUPG methods, considering different values of k in the velocity field. The same experimental results are given for the $\|e\|_{L^1(\Omega)}$ and $\|e\|_{L^2(\Omega)}$ norms. The time discretization is carried out with the second-order backward difference formula (BDF2).

6.0.1. Space convergence rates

The space convergence rates are evaluated by using $k = 2, 4, 6, 8$ in the velocity field; 500 and 2000 timesteps of sizes $\Delta t = 0.001h$; $\delta = h, 1$ for the $\|\tilde{e}\|_\delta$-norm, and three meshes with $h = 1/160, 1/320, 1/640$. Figs. 2 and 3 show the results for the CIP and SUPG methods. The numerical experiments show that for both methods, CIP and SUPG, the rates are between $\mathcal{O}(h^{\frac{1}{2}})$ and $\mathcal{O}(h^{\frac{3}{2}})$, satisfying an expression like

$$\|\tilde{e}\|_\delta \leq C_1(T)h^{3/2} + C_2(k, T)h^{1/2}, \tag{93}$$

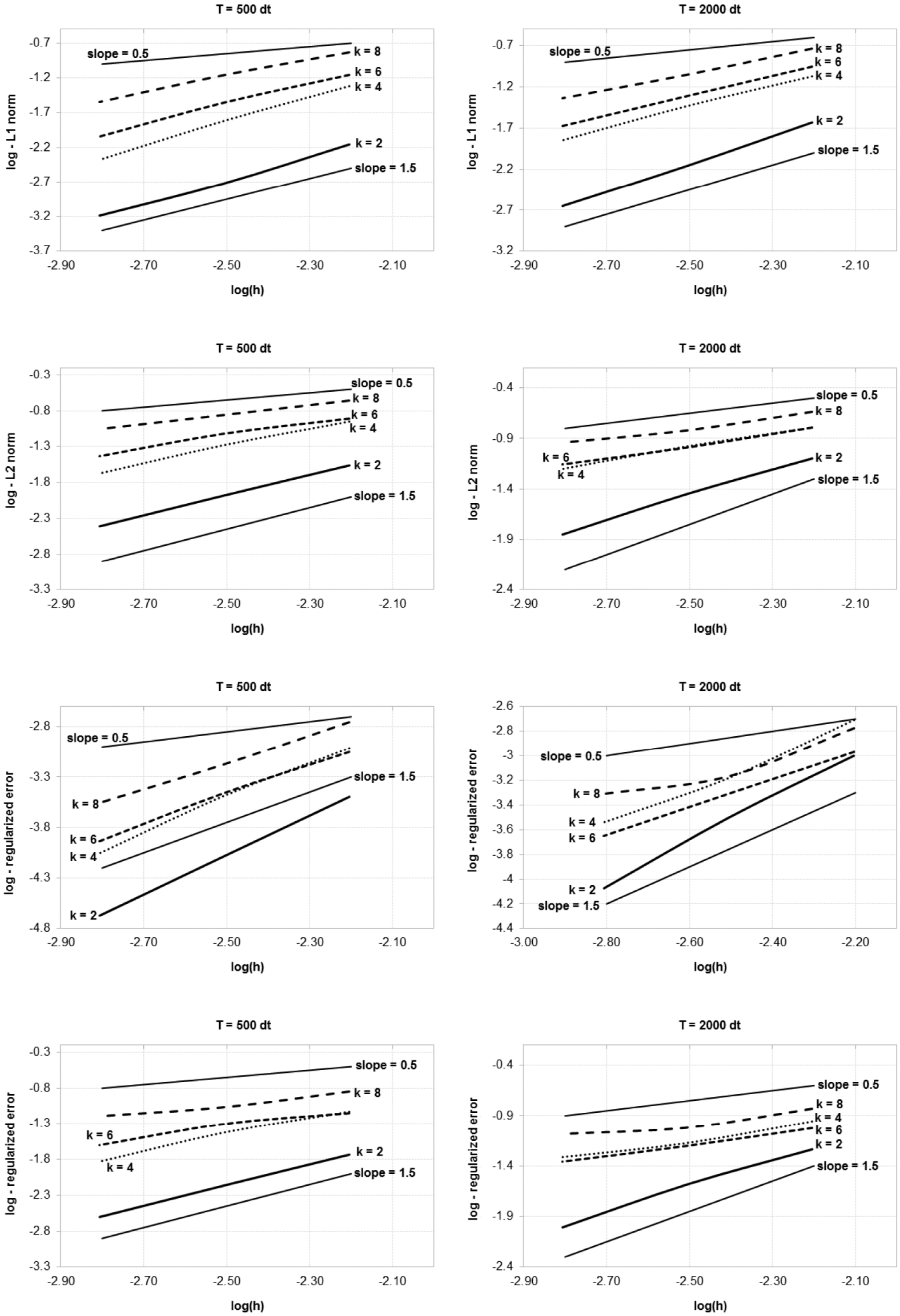


Fig. 2. Space convergence rates of the method CIP using the following norms: $\|\cdot\|_{L^1(\Omega)}$, $\|\cdot\|_{L^2(\Omega)}$ and $\|\tilde{e}\|_\delta$ with $\delta = 1$ and $\delta = h$, respectively.

so that, $C_1(T) \gg C_2(k, T)$ when T and k are small, and $C_1(T) \ll C_2(k, T)$ when T and k are big. Also, we have observed that

$$\|u - u_h\|_{L^1(\Omega)} \sim \|\tilde{e}\|_{\delta=1} \quad \text{and} \quad \|u - u_h\|_{L^2(\Omega)} \sim \|\tilde{e}\|_{\delta=h}.$$

The $\|\tilde{e}\|_\delta$-norm with $\delta = 1$ provided the smallest values in this experiment. For example, for the CIP method with $k = 6$ we have $\|\tilde{e}\|_{\delta=1} = \mathcal{O}(10^{-4})$ whereas the other norms are of order equal to or greater than of 10^{-2}. We can also observe that

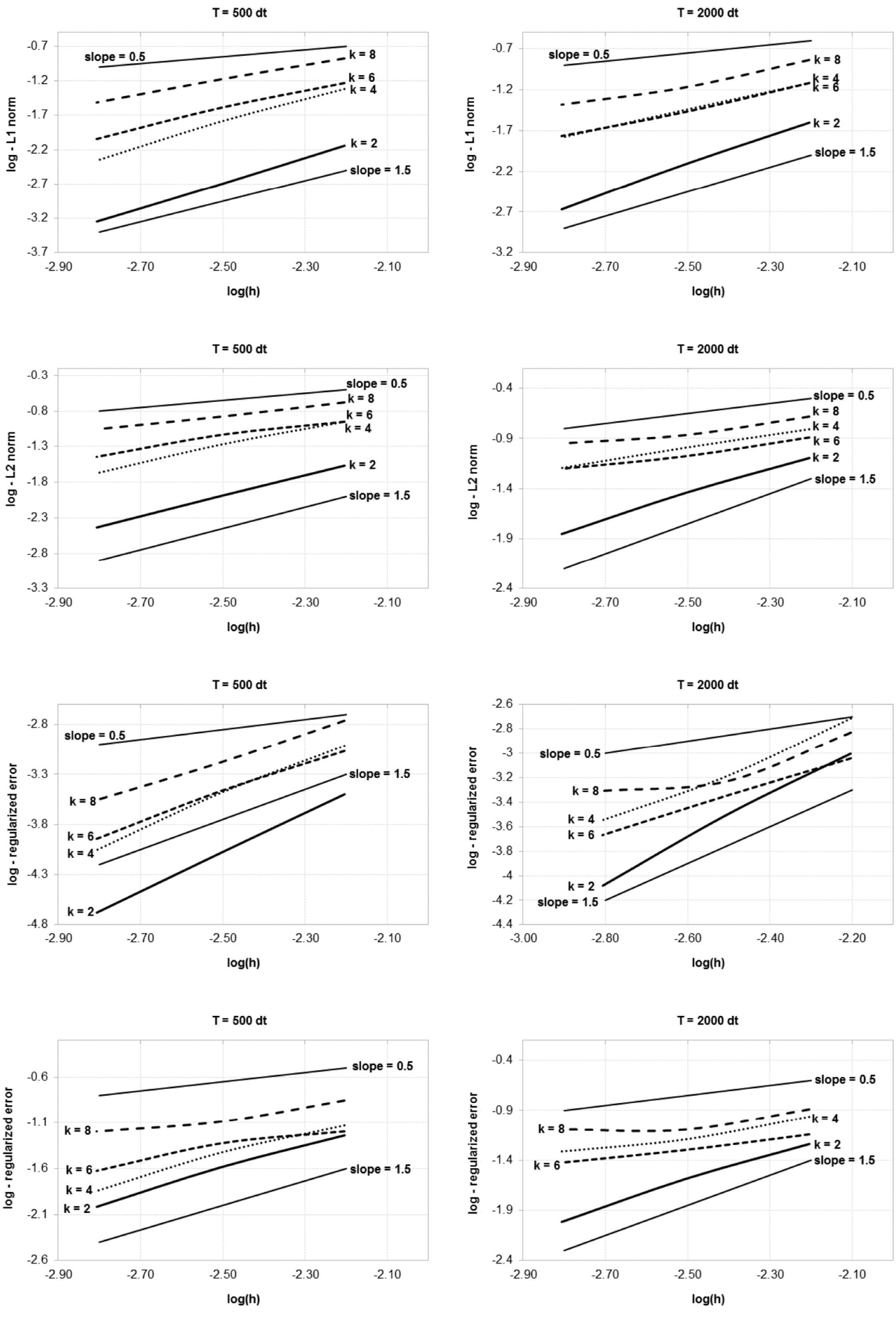


Fig. 3. Space convergence rates of the method SUPG using the following norms: $\|\cdot\|_{L^1(\Omega)}$, $\|\cdot\|_{L^2(\Omega)}$ and $\|\tilde{e}\|_\delta$ with $\delta = 1$ and $\delta = h$, respectively.

the convergence rate decays in time and for increasing values of k. Only for sufficiently long time (and using $k = 8$) the poor theoretical rate is observed.

6.0.2. Error growth in time

We have studied the growth of error in time for the CIP method over 5000 timesteps of size $\Delta t = 0.001h$ in two settings: one with $h = 1/100$ and the other with $kh = 1/50$ fixed. Similar results not reported here were obtained by

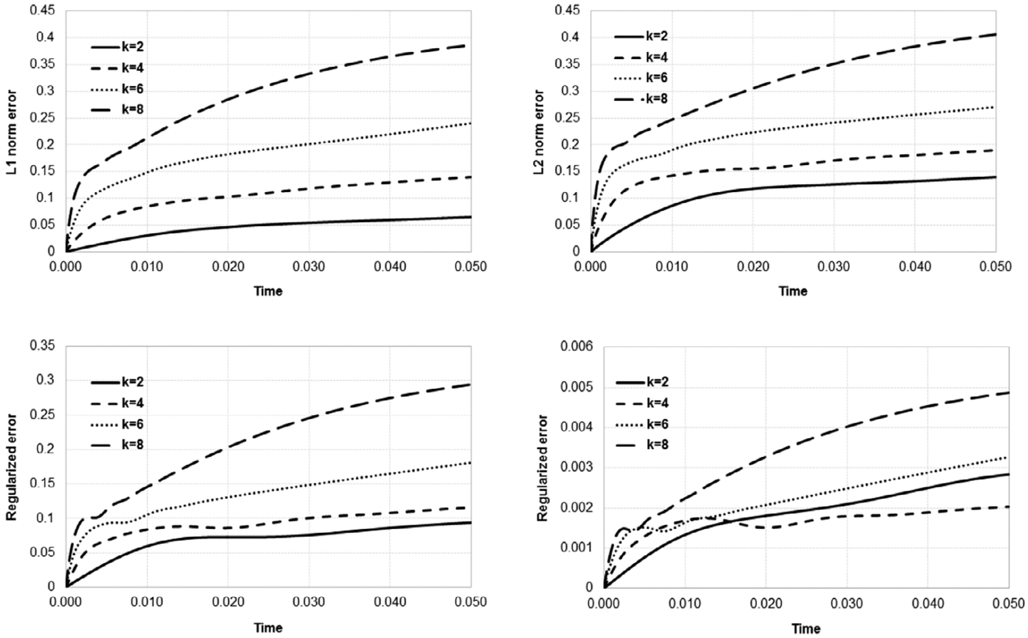


Fig. 4. Evolution in time of the error measured in the $\|\cdot\|_{L^1(\Omega)}$, $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{\delta=h}$ (bottom left) and $\|\cdot\|_{\delta=1}$ (bottom right) norms for CIP method using $k = 2, 4, 6, 8$ and $h = 1/100$.

the SUPG method. Fig. 4 shows the results obtained with $h = 1/100$ and different values of k. In general the error increases with increasing k as expected. The growth of the error in time is typically linear in the transient with slope $O(k^2)$. This is compatible with the exponential factor of our theorem, since $\|\nabla\boldsymbol{\beta}\|_{L^\infty(\Omega)} = O(k^2)$ and for small times $e^{c\|\nabla\boldsymbol{\beta}\|_{L^\infty(\Omega)}t} \sim 1+k^2 t$. Fig. 5 shows the inclination of the curves of the $\|\cdot\|_{L^2(\Omega)}$ error versus time, using $h = 1/100$ and different values of k. We have chosen $t = t^*$ satisfying $\|(u-u_h)(t^*)\|_{L^2(\Omega)} \cong 0.1$ and calculated the slope of the line formed by the points $(0, 0)$ and $(t^*, \|(u-u_h)(t^*)\|_{L^2(\Omega)})$, which is a linear approximation of the curve in the temporal interval $[0, t^*]$. The slopes obtained in terms of k, described by the slope function $s(\cdot)$, are of the order of k^2 and satisfy

$$s(k) = 7.7\left(\frac{k}{2}\right)^2 = \mathcal{O}(k^2), \quad k = 2, 4, 6, 8. \tag{94}$$

This means that

$$\|u-u_h\|_{L^2(\Omega)} \cong s(k)t = 7.7\left(\frac{k}{2}\right)^2 t, \quad \text{for } t \in [0, t^*],\ k = 2, 4, 6, 8. \tag{95}$$

As the velocities field satisfies $\|\nabla\boldsymbol{\beta}\|_{L^\infty(\Omega)} = \mathcal{O}(k^2)$, then we have

$$\|\nabla\boldsymbol{\beta}\|_{L^\infty(\Omega)} \sim s(k), \tag{96}$$

that is, there exists a constant $c_a > 0$ such that

$$\begin{aligned}
\|u-u_h\|_{L^2(\Omega)} &\le c_a\|\nabla\boldsymbol{\beta}\|_{L^\infty(\Omega)}t \\
&\le 1 + c_a\|\nabla\boldsymbol{\beta}\|_{L^\infty(\Omega)}t \\
&\le e^{c_a\|\nabla\boldsymbol{\beta}\|_{L^\infty(\Omega)}t}, \quad \text{for } t \in [0, t^*].
\end{aligned}$$

Similar results are obtained for $\|u-u_h\|_{L^1(\Omega)}$, $\|\tilde{e}\|_{\delta=h}$ and $\|\tilde{e}\|_{\delta=1}$. Fig. 6 presents the curves slope *versus* k for all norms. We can observe that the slopes increase with the order of k^2.

We have studied the growth of the error in time for $kh = 1/50$ fixed. The results are shown in Fig. 7 for several values of k. As the value of k increases the errors increase as well when the time is very small. The slopes of curves of error in terms of k (for a short time) were calculated as well for this case. Fig. 8 presents the curves slope *versus* k for all norms. The left figure shows the linear behavior ($\mathcal{O}(k)$) for $\|\tilde{e}\|_{\delta=1}$ norm, whereas in the right figure we see that the slopes are order k^2 (quadratic behavior).

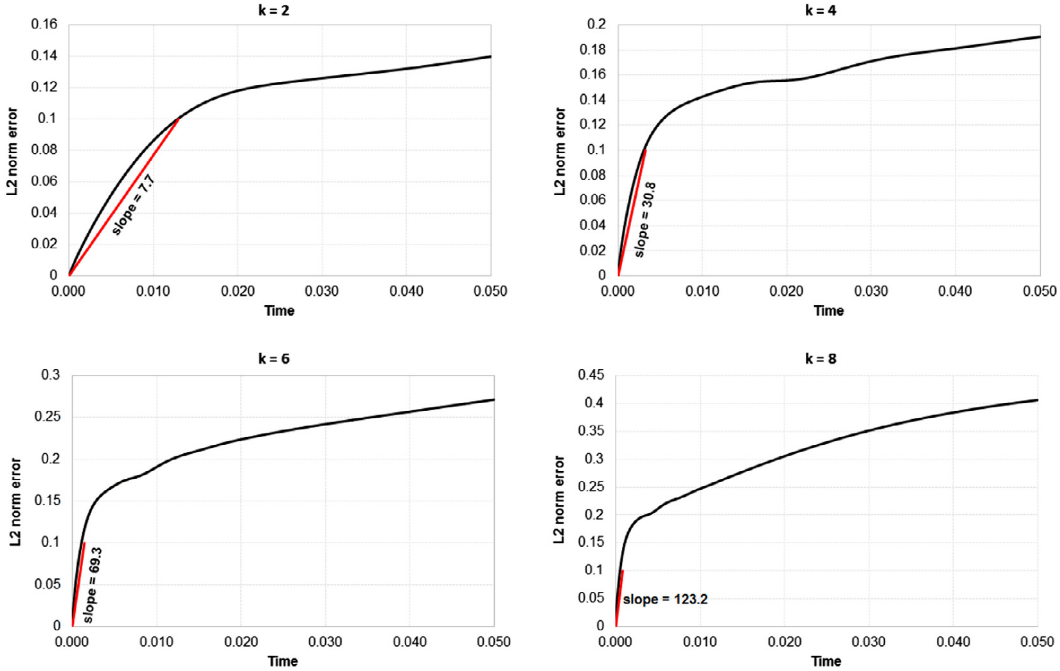


Fig. 5. Evolution in time of the L^2-norm error—CIP method, $h = 1/100$, $k = 2, 4, 6, 8$.

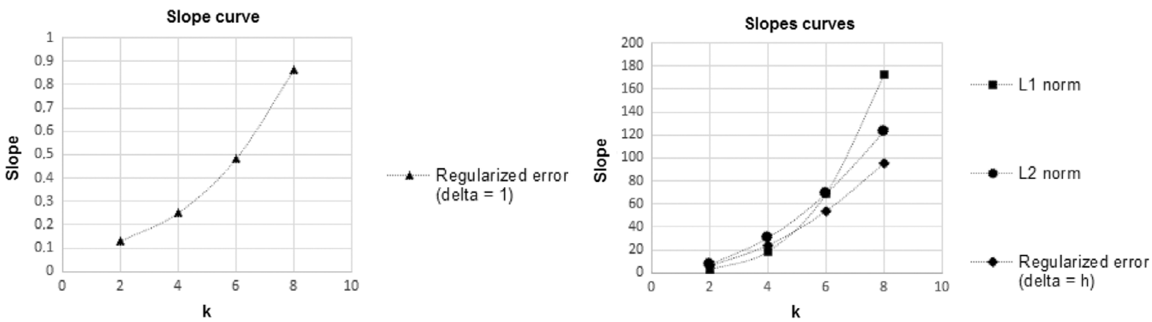


Fig. 6. CIP method—slope curves for all norms using $h = 1/100$.

7. Conclusions

We have discussed stabilized finite element methods for the transient advection–diffusion problem with high Péclet number with particular focus on the role of continuous dependence on data. We have proved error estimates for quantities that satisfy the continuous dependence assumption, using the enhanced control of the residual provided by the stabilization terms. A particular case that enters the framework is the weak norm estimates discussed in [1]. Indeed the required stability can be shown for a particular regularized error under the assumptions of two-scale decomposition of the velocity field. We have considered two stabilized methods: CIP and SUPG. In a numerical section we consider the special case of pure transport and showed that the convergence rate obtained in the estimate appears to be sharp. In this example the exponential growth with exponent proportional to the maximum velocity gradient was only observed when measuring the error in stronger norms than the one of the estimate, and for small times, giving some hope that the worst case scenario is not necessarily realized.

Acknowledgment

This research is partially supported by the Brazilian Government, through the Agency CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), under grant 18226-12-4.

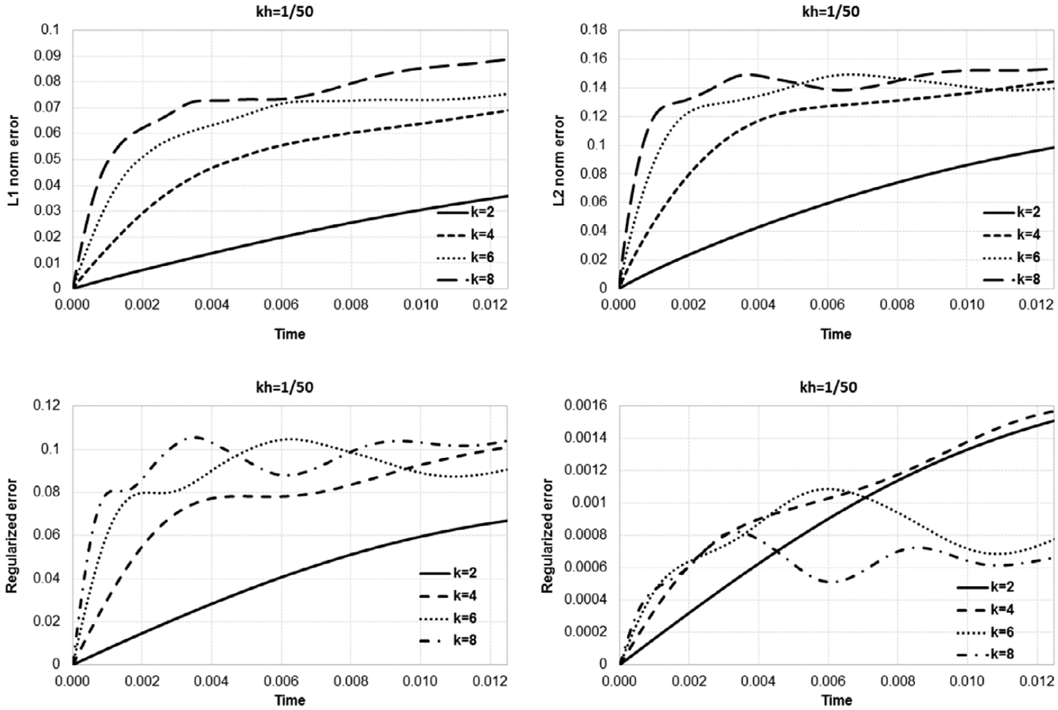


Fig. 7. Evolution in time of the error measured in the $\|\cdot\|_{L^1(\Omega)}$, $\|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_{\delta=h}$ (bottom left) and $\|\cdot\|_{\delta=1}$ (bottom right) norms for CIP method using $k = 2, 4, 6, 8$ and $kh = 1/50$ fixed.

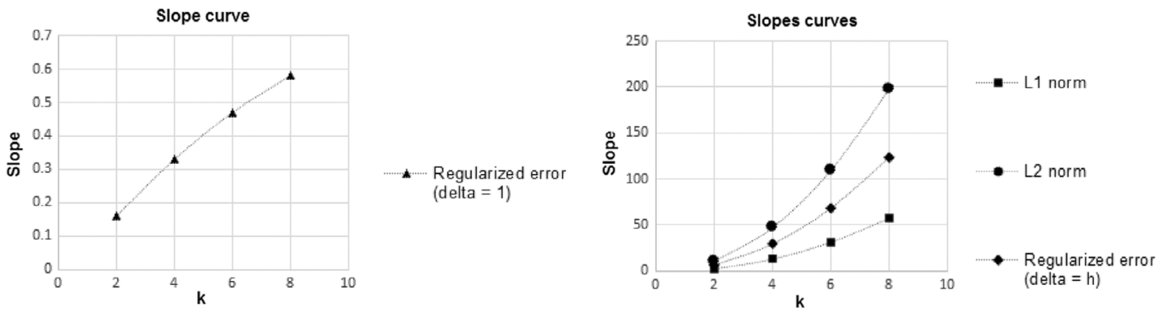


Fig. 8. CIP method—slope curves for all norms using $hk = 1/50$ fixed.

References

[1] E. Burman, Robust error estimates in weak norms for advection dominated transport problems with rough data, Math. Models Methods Appl. Sci. 24 (2014) 2663–2684.

[2] E. Burman, A. Ern, Stabilized Galerkin approximation of convection-diffusion-reaction equation: discrete maximum principle and convergence, Math. Comp. 252 (74) (2005) 1637–1652.

[3] A.N. Brooks, T.J.R. Hughes, Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations, Comput. Methods Appl. Mech. Engrg. 32 (1982) 199–259.

[4] C. Johnson, U. Nävert, J. Pitkäranta, Finite element methods for linear hyperbolic problems, Comput. Methods Appl. Mech. Engrg. 45 (1984) 285–312.

[5] J. Hoffman, On duality based a posteriori error estimation in various norms and linear functionals for LES, SIAM J. Sci. Comput. (26) (2004) 178–195.

[6] J. Hoffman, C. Johnson, A new approach to computational turbulence modeling, Comput. Methods Appl. Mech. Engrg. 195 (23) (2006) 2865–2880.

[7] Erik Burman, Error estimates for stabilized finite element methods applied to ill-posed problems, C. R. Math. Acad. Sci. Paris (ISSN: 1631-073X) 352 (7–8) (2014) 655–659. http://dx.doi.org/10.1016/j.crma.2014.06.008.

[8] E. Burman, Robust error estimates for stabilized finite element approximations of the two dimensional Navier-Stokes' equations at high Reynolds number, Comput. Methods Appl. Mech. Engrg. 288 (2015) 2–23.

[9] R. Verfürth, Robust a posteriori error estimates for nonstationary convection-diffusion equations, SIAM J. Numer. Anal. (ISSN: 0036-1429) 43 (4) (2005) 1783–1802. http://dx.doi.org/10.1137/040604273 (electronic).

[10] Alexandre Ern, Jennifer Proft, A posteriori discontinuous Galerkin error estimates for transient convection-diffusion equations, Appl. Math. Lett. (ISSN: 0893-9659) 18 (7) (2005) 833–841. http://dx.doi.org/10.1016/j.aml.2004.05.019.

[11] Dmitri Kuzmin, Sergey Korotov, Goal-oriented a posteriori error estimates for transport problems, Math. Comput. Simulation (ISSN: 0378-4754) 80 (8) (2010) 1674–1683. http://dx.doi.org/10.1016/j.matcom.2009.03.008.

[12] Vít Dolejší, Alexandre Ern, Martin Vohralík, A framework for robust a posteriori error control in unsteady nonlinear advection–diffusion problems, SIAM J. Numer. Anal. (ISSN: 0036-1429) 51 (2) (2013) 773–793. http://dx.doi.org/10.1137/110859282.

[13] V. John, J. Novo, A robust SUPG norm a posteriori error estimator for stationary convectiondiffusion equations, Comput. Methods Appl. Mech. Engrg. 255 (1) (2013) 289–305.
[14] L. Tobiska, R. Verfürth, Robust a posteriori error estimates for stabilized finite element methods, IMA J. Numer. Anal. 35 (4) (2015) 1652–1671.
[15] P. Clément, Approximation by finite element functios using local regularization, RAIRO Anal. Numer. R-2 (9) (1975) 77–84.
[16] J. Douglas, T. Dupont, Interior penalty procedures for elliptic and parabolic Galerkin methods, in: Computing Methods in Applied Sciences, in: Lecture Notes in Phys., vol. 58, Springer-Verlag, Berlin, 1976.
[17] E. Burman, G. Smith, Analysis of the space semi-discretized SUPG method for transient convection-diffusion equations, Math. Models Methods Appl. Sci. 21 (10) (2011) 2049–2068.